Derived TQFT's Quantum Mechanics

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Introduction

Most mathematical constructions of topological quantum field theories (TQFT's) are build using constructions from classical algebra e.g. vector spaces. This captures a large classes of examples appearing in physics e.g. modular categories or fusion categories. Nevertheless many others can only be described mathematically by using homological algebra. Examples of those are topological twisting of supersymmetric theories (e.g. Donaldson-theory) or categories of D-branes in A and B-models. In the following we describe how homological algebra influences 1-dimensional TQFT's.

Quantum Mechanics as a 1d-TQFT

Consider a Quantum mechanical system and denote by V the vector space of ground states. We denote such a state by $|\Psi\rangle \in V$. Dually we have the dual state $\langle\Psi| \in V^\vee$. To describe the Quantum mechanics of ground states we introduce a graphical calculus. In the graphical calculus denote V by " + " and $V^\vee$ by " − ". Vertically aligned dots denote the tensor product of the corresponding vector spaces. These correspond to non-interacting systems.

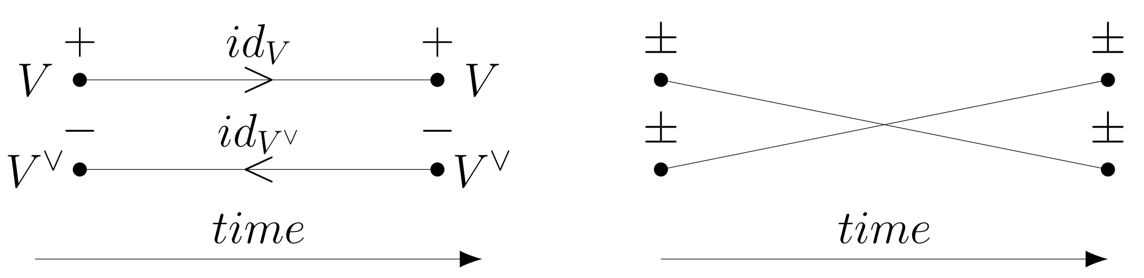

Figure 1. a) Evolution of states b) Change of order $|\Psi\rangle|\Phi\rangle \mapsto |\Phi\rangle|\Psi\rangle$

The time evolution of a state is described by directed lines. We associate to those linear maps from the state spaces on the left to those on the right. To a horizontal line we associate the identity map.
Further we can pair states $|\Psi\rangle \in V$ wit dual sates $\langle\Phi| \in V^\vee$ to obtain transition amplitudes. Under the assumption that V is finite dimensional it has a basis $\{|\Psi_i\rangle\}_{i\in I}$ and we can additionally define Figure 2 b)

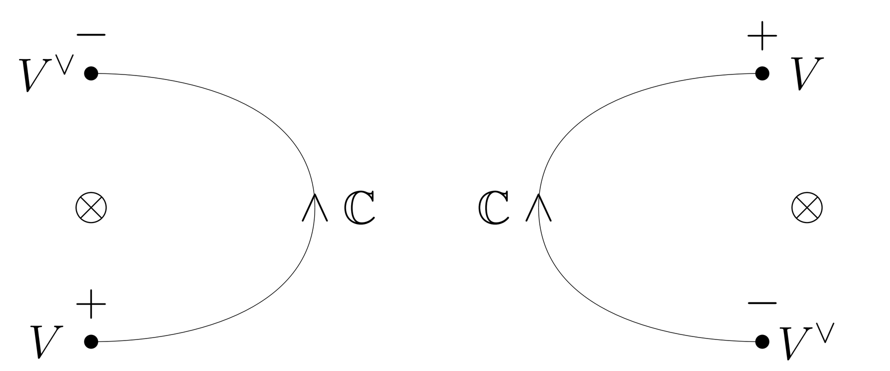

Figure 2. a) Evaluation $\langle\Psi|\Phi\rangle$ b) Creation of identity $\sum_{i\in I}|\Psi_i\rangle\langle\Psi_i|$

These are the basic building blocks we can use to understand more difficult diagrams. Since the ground states have no dynamics the interpretation of the graphical calculus is independent of the length and shape of lines as long as we keep the endpoints fixed. Therefore we have an identification

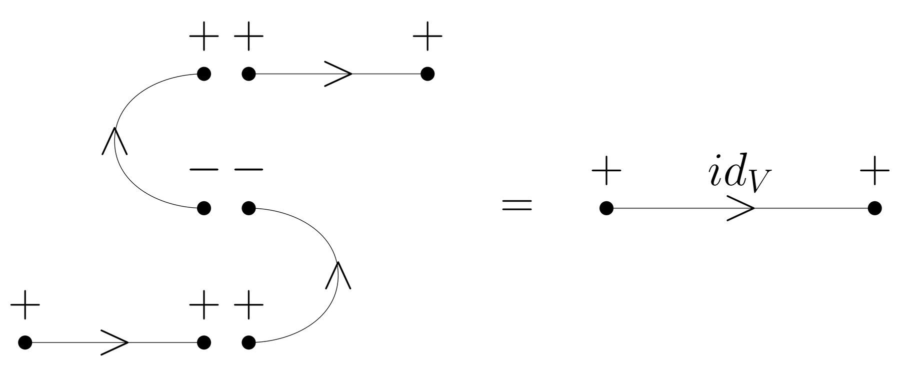

Figure 3. Graphical presentation of $\sum_{i\in I}|\Psi_i\rangle\langle\Psi_i|\Phi\rangle = |\Phi\rangle$

This is the fundamental consistency condition that our graphical calculus has to satisfy to be well defined. An interesting diagram that we can build is the circle

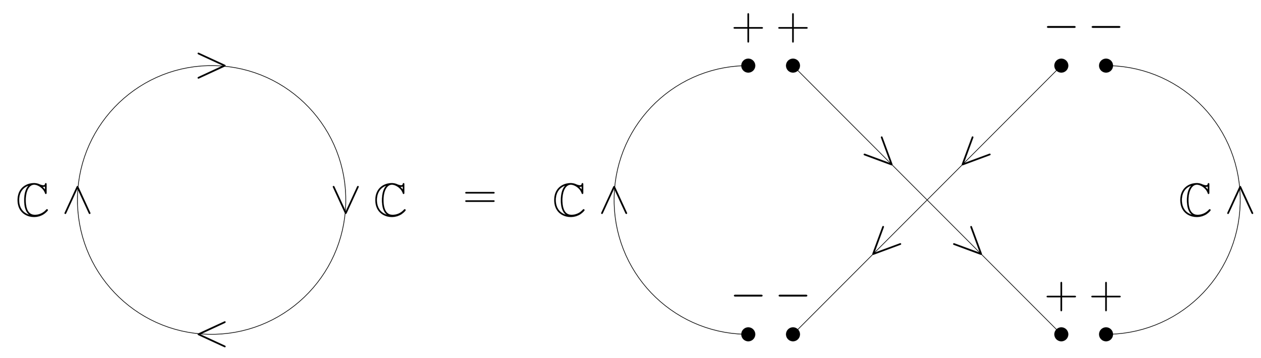

Figure 4. Construction of the circle

The value of the circle is given by

$$\sum_{i\in I}\langle\Psi_i|\Psi_i\rangle = dim(V) \tag{1}$$

A consistent graphical calculus as above is equivalent to a 1-dimensional TQFT. This calculus is consistent, if and only if the vector space V is finite dimensional. Can we construct a graphical calculus more generally?

Homological algebra

We consider a possible ∞-dimensional vector space V, with a $\mathbb{Z}$-grading $V \simeq \oplus_{i\in\mathbb{Z}} V_i$ and with linear maps $\{d_i\}_{i\in\mathbb{Z}}$ of the form

$$\dots \xrightarrow{d_{i+3}} V_{i+2} \xrightarrow{d_{i+2}} V_{i+1} \xrightarrow{d_{i+1}} V_i \xrightarrow{d_i} V_{i-1} \xrightarrow{d_{i-1}} \dots \tag{2}$$

s.th. $d_i \circ d_{i+1} = 0$. We call such an object a complex of vector spaces and denote it by $(V_\bullet, d_\bullet)$.
Since $d_i \circ d_{i+1} = 0$ it follows, that $im(d_{i+1}) \subset ker(d_i)$. We call the quotient

$$H_i(V_\bullet) = ker(d_i)/im(d_{i+1}) \subset V_i \tag{3}$$

the i-th homology space of the complex $V_\bullet$ and the direct sum of all of these $H_\bullet(V)$ the homology of $V_\bullet$.
Analogous to the case of vector spaces every complex $V_\bullet$ admits a dual complex $V_\bullet^\vee$. A chain map $f_\bullet : (V_\bullet, d_\bullet^V) \to (W_\bullet, d_\bullet^W)$ is given by a collection of linear maps $\{f_i : V_i \to W_i\}_{i\in\mathbb{Z}}$, s.th. $d_i^W \circ f_i = f_{i-1} \circ d_i^V$.
The interesting thing for morphisms of chain complexes is that they admit a weaker notion then equality. Two morphisms of chain complexes $f_\bullet, g_\bullet \; : \; V_\bullet \to W_\bullet$ are called chain homotopic, if there exists a collection of maps $\{H_i : V_i \to W_{i+1}\}_{i\in\mathbb{Z}}$, s.th.

$$d_{i+1}^W H_i + H_{i-1} d_i^V = f_i - g_i \tag{4}$$

In homological algebra we can therefore define two chain maps to be "the same", if they are chain homotopic.

Derived 1d-TQFT

We now use homological algebra to enhance our graphical calculus. Consider a complex $V_\bullet$ and assume that $\mathbf{H}_\bullet(\mathbf{V})$ is finite dimensional. We choose a basis $\{|\Phi_i\rangle\}_{i\in I}$ of $H_\bullet(V)$ with dual basis $\{\langle\Phi_i|\}_{i\in I}$. In the construction of our graphical calculus we can proceed as before, but we associate to Figure 2. b) the map

$$\lambda \in \mathbb{C} \mapsto \lambda \sum_{i\in I}|\Phi_i\rangle\langle\Phi_i| \in V_\bullet \otimes V_\bullet^\vee \tag{5}$$

Using this interpretation we can compute the consistency condition Figure 3 to be the chain map

$$|\Psi\rangle \in V_i \mapsto \sum_{i\in I}\langle\Phi_i|\Psi\rangle|\Phi_i\rangle \in H_i(V_\bullet) \subset V_i \tag{6}$$

that projects onto the homology. This is map is not equal to the identity, but chain homotopic! Therefore if we consider the value of the graphical diagrams up to chain homotopy the graphical calculus becomes consistent.
Similarly we can compute the value of the circle to be

$$\sum_{i\in I}\langle\Phi_i|\Phi_i\rangle = \sum_{i\in I} dim(H_i(V)) \tag{7}$$

Generalizing the case of vector spaces the following is true.

Classification of derived 1-dimensional TFT's [1]

Let $V_\bullet \in \mathbf{D}(\mathbb{C})$ be a chain complex. $V_\bullet$ defines a 1-dimensional oriented TFT

$$\mathbf{Z} : \mathbf{Bord}_1^{or} \to \mathbf{D}(\mathbb{C}) \tag{8}$$

with values in $\mathbf{D}(\mathbb{C})$ the derived category of vector spaces, iff $V_\bullet$ has finite dimensional homology. In particular every finite dimensional vector space defines such a TFT.

Outlook

A similar story is known in the study of 2-dimensional fully extended TQFT's [2]. Derived 2d TQFT's are classified by smooth and proper dg algebras, a more general class containing semi-simple algebras. My research tries to understand this kind of phenomena also in higher dimensions, in particular in dimension 3. These kind of TQFT's are generalizations of so called Turaev-Viro TQFT's. Interesting questions in this direction are:

- What is the correct generalization of a fusion category?
- Which non-semi simple tensor categories induce such TQFT's?
- How to compute the TQFT?

References

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